<u>Lecture 1b</u>

Review on Math (continued)

$s \mathbin{⩤} r$ domain subtraction

s → set

r → relation

overridden by

$r \mathbin{\lhd\!\!\!+} t$ overriding

r → relation

t → relation

$r = \{(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)\}$

Definition: $r \lhd t = \{\, (d, r) \mid \underbrace{(d, r) \in t}_{\text{relation}} \overset{\text{union}}{\vee} \underbrace{((d, r) \in r \wedge d \notin \mathrm{dom}(t))}_{\text{another relation}} \,\}$

e.g.,

$r \lhd \{(a, 3), (c, 4)\}$

$$r \lhd \{\underbrace{(a, 3), (c, 4)}_{\mathrm{dom}(t) = \{a, c\}}\}$$

$$\{(a, 3), (c, 4)\} \cup \{(b, 2), (b, 5), (d, 1), (e, 2), (f, 3)\}$$

$$= \{\ \underline{\qquad\qquad\qquad\qquad}\ \}$$

$r = \{(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)\}$

$$\boxed{r[s] = \operatorname{ran}(s \lhd r)}$$

$$r[\underbrace{\{a, b\}}_{s}] = \operatorname{ran}(\underline{\{a, b\} \lhd r})$$

$$= \operatorname{ran}(\{(a, 1), (b, 2), (a, 4), (b, 5)\})$$

$$= \{1, 2, 4, 5\}$$

Side note: databases

↳ relational databases (SQL queries)

↳ relational algebra

$r = \{(a, 1), (b, 2), (c, 3), (a, 4), (b, 5), (c, 6), (d, 1), (e, 2), (f, 3)\}$

$r \mathbin{\triangleleft\!\!+} t = t \cup (\mathrm{dom}(t) \mathbin{⩤} r)$ → algebraic property.

$a + b = b + a$

$$r \mathbin{\triangleleft\!\!+} \underbrace{\{(a,3),(c,4)\}}_{t}$$

$$r \mathbin{\triangleleft\!\!+} \{\underbrace{(a,3),(c,4)}_{t}\}$$

$$= \{(a,3),(c,4)\} \cup (\{a,c\} \mathbin{⩤} r)$$

$\{(b,2),(b,5),(d,1),(e,2),(f,3)\}$

$= \{\_\_\_\_\_\_\_\_\_\_\}$

$$\text{isFunctional}(r) \iff \forall s, t_1, t_2 \bullet (s \in S \wedge t_1 \in T \wedge t_2 \in T) \Rightarrow \boxed{((s, t_1) \in r \wedge (s, t_2) \in r \Rightarrow t_1 = t_2)}$$

to disprove → find witness (satisfying antecedent but violating consequence)

||| contrapositive $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$

$$t_1 \neq t_2 \Rightarrow (s, t_1) \notin r \vee (s, t_2) \notin r$$

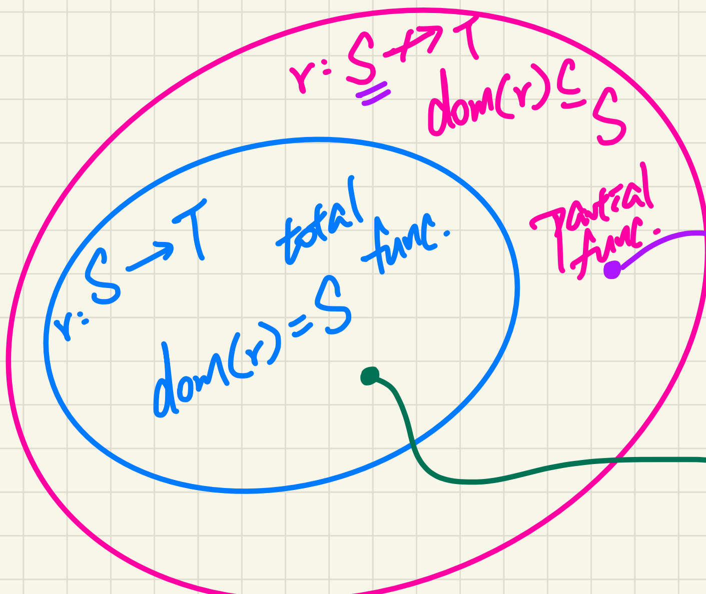
What is the smallest relation satisfying the functional property?

↳ $\emptyset$ ∵ we cannot find any witness to disprove that it violates the functional property ∴ $\emptyset$ is a function

$\text{dom} = \{2, 1\} \subseteq \subset$ $\qquad$ $\text{dom} = \{2, 3, 1\} \subseteq \subset \times$ ✓✓✓

e.g., $\{\ \underbrace{\{(\mathbf{2}, a), (\mathbf{1}, b)\}}_{\text{function}}, \underbrace{\{(\mathbf{2}, a), (\mathbf{3}, a), (\mathbf{1}, b)\}}_{\text{function}}\ \} \boxed{\subseteq} \{1, 2, 3\} \nrightarrow \{a, b\}$

$\nrightarrow$: the set of possible partial functions



partial, not total

$s \in S$ s.t. $r(s)$ undefined

total & partial

	injective	surjective	bijective
partial			✗
total			

Injective Functions

$\texttt{isInjective}(f)$

$\Longleftrightarrow$

$\forall s_1, s_2, t \bullet (s_1 \in S \land s_2 \in S \land t \in T) \Rightarrow ((s_1, t) \in f \land (s_2, t) \in f \Rightarrow s_1 = s_2)$

$b = b \Rightarrow 1 = 3$ False

If f is a ***partial injection***, we write: $\boxed{f \in S \rightarrowtail\!\!\!\!\!\!+ \; T}$ $\rightarrowtail\!\!\!\!\!\!+\!\!\longrightarrow$

- e.g., $\{\emptyset, \{(1,\mathbf{a})\}, \{(2,\mathbf{a}), (3,\mathbf{b})\}\} \subseteq \{1,2,3\} \rightarrowtail\!\!\!\!\!\!+ \{a,b\}$
- e.g., $\{(1,\mathbf{b}), (2,a), (3,\mathbf{b})\} \notin \{1,2,3\} \rightarrowtail\!\!\!\!\!\!+ \{a,b\}$
- e.g., $\{(1,\mathbf{b}), (3,\mathbf{b})\} \notin \{1,2,3\} \rightarrowtail\!\!\!\!\!\!+ \{a,b\}$

partial, not inj.

$\emptyset$: 1. not total
2. injective
∵ no witnesses of violation

If f is a ***total injection***, we write: $\boxed{f \in S \rightarrowtail T}$ $\rightarrowtail\!\!\longrightarrow$

- e.g., $\{1,2,3\} \rightarrowtail \{a,b\} = \emptyset$ → $\{(1,a),(2,b),(3,a)\}$
- e.g., $\{(2,d), (1,a), (3,c)\} \in \{1,2,3\} \rightarrowtail \{a,b,c,d\}$
- e.g., $\{(2,d), (1,c)\} \notin \{1,2,3\} \rightarrowtail \{a,b,c,d\}$ → not total, inj
- e.g., $\{(2,\mathbf{d}), (1,c), (3,\mathbf{d})\} \notin \{1,2,3\} \rightarrowtail \{a,b,c,d\}$ → total, not inj.

the set of all possible total injections

$(2,d), (3,d)$

$d = d \Rightarrow 2 = 3$ False.

Surjective Functions

$$\mathit{isSurjective}(f) \iff \mathrm{ran}(f) = T$$

If f is a ***partial surjection***, we write: $\boxed{f \in S \twoheadrightarrow\!\!\!\!\!\!+\;\; T}$ $\quad$ $+\!\!\twoheadrightarrow$

- e.g., $\{\{(1,\mathbf{b}),(2,\mathbf{a})\},\{(1,\mathbf{b}),(2,\mathbf{a}),(3,\mathbf{b})\}\} \subseteq \{1,2,3\} \twoheadrightarrow\!\!\!\!\!\!+\;\; \{a,b\}$
- e.g., $\{(2,\mathbf{a}),(1,\mathbf{a}),(3,\mathbf{a})\} \notin \{1,2,3\} \twoheadrightarrow\!\!\!\!\!\!+\;\; \{a,b\}$ ran = {a} partial, not sur.
- e.g., $\{(2,\mathbf{b}),(1,\mathbf{b})\} \notin \{1,2,3\} \twoheadrightarrow\!\!\!\!\!\!+\;\; \{a,b\}$ ran = {b} partial, not sur.

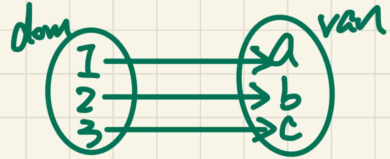
not inj.

If f is a ***total surjection***, we write: $\boxed{f \in S \twoheadrightarrow T}$ $\quad$ $\twoheadrightarrow$

- e.g., $\{\{(2,a),(1,b),(3,a)\},\{(2,b),(1,a),(3,b)\}\} \subseteq \{1,2,3\} \twoheadrightarrow \{a,b\}$ total, sur.
- e.g., $\{(\mathbf{2},a),(\mathbf{3},b)\} \notin \{1,2,3\} \twoheadrightarrow \{a,b\}$ dom = {2,3} not total, sur.
- e.g., $\{(2,\mathbf{a}),(3,\mathbf{a}),(1,\mathbf{a})\} \notin \{1,2,3\} \twoheadrightarrow \{a,b\}$ ran = {a}

total, not sur.

Bijective Functions



f is ***bijective***/***a bijection***/*one-to-one correspondence* if f is ***total***, ***injective***, and ***surjective***.

$\rightarrowtail\!\!\!\!\twoheadrightarrow$

- e.g., $\{1,2,3\} \rightarrowtail\!\!\!\!\twoheadrightarrow \{a,b\} = \emptyset$ $\{(1,\overset{a}{a}),(2,b),(3,?)\}$
- e.g., $\{\{(1,a),(2,b),(3,c)\},\{(2,a),(3,b),(1,c)\}\} \subseteq \{1,2,3\} \rightarrowtail\!\!\!\!\twoheadrightarrow \{a,b,c\}$
- e.g., $\{(\mathbf{2},b),(\mathbf{3},c),(\mathbf{4},a)\} \notin \{1,2,3,4\} \rightarrowtail\!\!\!\!\twoheadrightarrow \{a,b,c\}$ not total, inj, sur.
- e.g., $\{(1,\mathbf{a}),(2,b),(3,c),(4,\mathbf{a})\} \notin \{1,2,3,4\} \rightarrowtail\!\!\!\!\twoheadrightarrow \{a,b,c\}$ total, not inj, sur.
- e.g., $\{(1,\mathbf{a}),(2,\mathbf{c})\} \notin \{1,2\} \rightarrowtail\!\!\!\!\twoheadrightarrow \{a,b,c\}$ ran = $\{a,c\}$

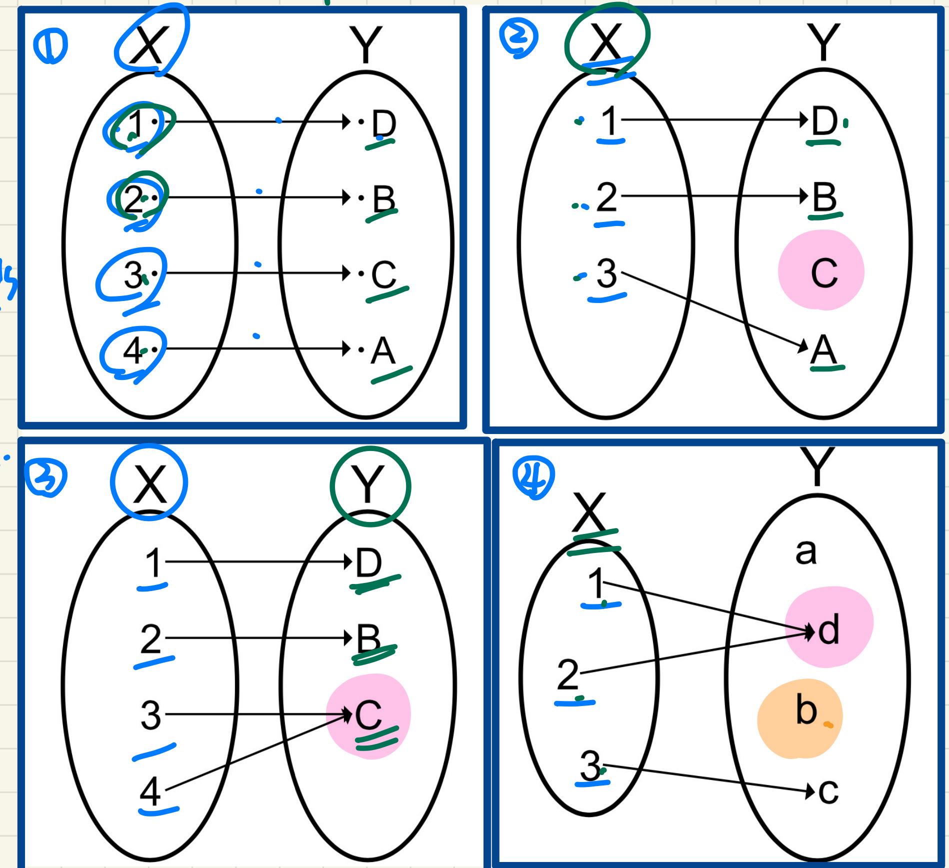
total ✓

inj. ✓

sur. ✗

Exercise

$dom(①) = X$
$ran(①) = Y$



✓ Exercise

Make a function that's partial but not total.

	①	②	③	④
partial	✓	✓	✓	✓
total	✓	✓	✓	✓
inj.	✓	✓	✗	✗
sur.	✓	✗	✓	✗
bij.	✓	✗	✗	✗

Formalizing Arrays as Functions

Not partial inj.

0	1	2
alan	jim	alan

$a = \{(0, alan), (1, jim), (2, alan)\}$

programming

indices ≈ domain

String[] a = new String[5];

0	1	2	3	4
alan	mark	tom	jim	jonathan

content ≈ range

Strings 0 "" 1 " " 2 "_ _"

$a = \{(0, \text{"alan"}), (1, \text{"mark"}), (2, \text{"tom"}), (3, \text{"jim"}), (4, \text{"jona."})\}$

formalization in math.

Should a be formalized/modeled as a relation? $\mathbb{Z} \leftrightarrow String$

No. ∵ $\{(0, alan), (0, jim)\}$ → in reality, only one element may be stored at each index

Partial Surjection $\mathbb{Z} \twoheadrightarrow \mathbb{Z}$ (int, 32 bits)

0	1	…	a.length −1
		…	